Computation of Secular Perturbations. By R. T. A. Innes.

I.

It is well known that the Lagrangian equations for the variation of a planet's elements (the planet being considered of infinitesimal dimensions) are exact. It is in the integration of these equations that difficulties arise. To avoid these difficulties several assumptions are made, such as that the planets can be considered in pairs (*i.e.* the problem of three bodies) and do not react on each other; that the elements on the right-hand side of the equations are constant, and that the perturbative function can be developed in a converging series. When we are concerned with the mutual perturbations of the eight major planets, the above assumptions are sufficiently good; the errors introduced by the first two assumptions are eliminated later by taking into account the powers of the masses higher than the first, the third assumption is nearly justified by the smallness of the eccentricities and mutual inclinations of the eight major planets. In cases of difficulty, the Lagrangian equations are abandoned and the perturbations found by other processes; thus Hill found that the Lagrangian method as used by Le Verrier for the theory of the motion of Jupiter and Saturn was insufficient, so that in his theory he adopted the processes developed by Hansen.

Gauss showed that the secular part of the perturbations (to the first power of the masses) could be found without using a development of the perturbative function. The rationale of Gauss's method is as follows:—Let us imagine that the perturbative function is developed in a series as follows,

$$P_0 = P_1 + P_s \sin M + P_c \cos M + P_{2s} \sin 2M + \text{etc.}$$

where the coefficients P contain, besides certain quantities depending on the elements of the disturbed planet, the elements and position of the disturbing body. Multiplying each side by dM and integrating around the circumference gives

$$\frac{1}{2\pi}\int_0^{2\pi} P_0 dM = P_1$$

Then writing $P_1 = p_1 + p_s \sin M_1 + p_c \cos M_1 + \text{etc.}$, where the coefficients p depend only on the elements of both planets and do not contain the time explicitly, a second integration gives

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} P_0 dM dM_1 = p_1$$

It is on the quantity p_1 that the secular perturbations depend. In the general case it is impossible to find this quantity by algebraical expansions. Gauss proved that one of these integrations could be

reduced to elliptic integrals; the second has to be a mechanical integration. Gauss, as is well-known, gave a geometrical title to his paper and did not prepare his formulæ for numerical application. Bour and others have given further geometrical interpretations of Gauss's formulæ, which, interesting as they are, call for no attention here.

Since Dr. G. W. Hill published his first paper on Gauss's Method of Computing the Secular Perturbations of the Planets (Astron. Papers prepared for the use of the American Ephemeris, 1882; Collected Works, T. ii., 1906) many numerical applications of this method have been made. Hill did not leave the method in final shape, for no sooner had he finished his paper than he proceeded to give a substantial modification in an addendum by making use of Gauss's arithmetico-geometrical mean. Both of Hill's solutions require extensive tables of the elliptic integrals for their easy application. Hill provided these tables for his first solution, and Callandreau and the writer did the same for Hill's second method. Previously to Hill's work there had been but little use made of Gauss's method. We learn from Bode's *Astronomisches Jahrbuch*, 1819, p. 229, that before Gauss had actually published his memoir, Nicolai had computed the sec. var. of the elements of the Earth's orbit by this method, but no details of the work were given. The late Professor J. C. Adams applied the method to the perturbation of the orbit of the November Meteors (see *M. N.*, 1867, Apl.; Collected Works, T. ii. pp. 194–200). It is well known that the method as expounded by Gauss requires the solution of a cubic equation, but Adams avoids the cubic by neglecting a portion of the terms factored by the square of the eccentricity of the disturbing body, and so reduces the equation to a quadratic. This is no doubt the modification which Adams stated greatly facilitates the application of Gauss's formulæ. As a practical step this modification was quite justified, but it takes away from the interest of the problem, and, as will appear immediately, is of no real advantage. Halphen, in 1886 (C.R., T. ciii.) announced that the problem could be completely solved without the labour of obtaining the roots of the cubic; he gave an exposition of his method in his *Fonctions Elliptiques*, 1888, T. ii. I was unable to follow Halphen's exposition, to which Tisserand's explanation added nothing (*Mec. Cel.*, T. i.), and it was only on obtaining by chance a copy of Dr Louis Arndt's paper on Halphen's solution (*Recherches . . . . Pert. Seculaires*, Neuchâtel, 1896) that the solution was grasped. The formulæ for numerical application given by Arndt assume that g_3 is always positive, so that his solution will only apply to the easier cases, although Halphen drew special attention to the case of g_3 being negative. Arndt has provided tables for the cases in which Legendre's modular angle θ will not exceed 45°. Hill has again revised his solution, and shows that by using Jacobi's nome q tables are practically unnecessary, but the roots of the cubic are still required (*Ast. Jour.*, No. 511, 1901).

As already stated, Gauss's method is rigorous to the first power

of the masses. It has been applied to most of the mutual actions of the major planets, but unnecessarily so in many cases, as the tables given by Newcomb and Le Verrier are ample and much simpler to use unless the ratio of the mean distances exceeds one-half; in other words, the small inclinations and eccentricities of the orbits of the major planets permit algebraical expansion, which only becomes long when this ratio exceeds one-half.

This method has its rôle in computing the first order secular perturbations of the periodical comets and minor planets. An interesting application to Eros has recently been made by Herr W. Dziewulski (*Säkulare Marsstörungen . . . des Eros*, Cracovie, 1906). He shows incidentally that this method gives better results in the case of Eros than the Lagrangian method, which takes into account all powers of the masses, but not of the inclinations and eccentricities. In fact, the Lagrangian approximation will in some cases not even give the correct signs, far less the correct figures. Tolerable approximations are only given in the cases of the planets with preponderating masses.

Mr C. J. Merfield, of Sydney, has recently announced the completion of a computation of the secular variation of Eros.

In this paper Hill's first exposition has been adopted as a working basis, and in the introduction of the elliptic functions Schwarz's notation, as developed in Padé's translation (*Formules et Propositions*, Paris, 1894), has been followed, except that the quantity h is replaced by Jacobi's letter q. No proofs of formulæ found in previous papers are given.

The aim throughout has been simplicity in numerical application. The final formulæ have been framed for that end, and have been submitted to numerical tests. The pure mathematician dismisses one with a series guaranteed to be absolutely convergent for certain values of the argument, but a trial often proves that the series, though it may be absolutely convergent, is arithmetically absolutely useless. It will appear that if an accuracy of six places of figures suffices, the use of series can be entirely evaded, a result by no means to be expected *a priori*.

The two main points in this paper are (1) the simplification of the formulæ by the introduction of the Weierstrassian cubic; (2) the new series for the computation of the periods of the elliptic functions,—series which are remarkably inert, nearly all the variation being thrown on to a simple trigonometrical factor of the series.

Thus the computer has the choice of two main methods in applying Gauss's principle,—(1) solving the cubic and using Jacobi's nome q (Hill, *Ast. Jour.*, No. 511); or (2) using the invariants of the cubic and two hypergeometrical series. Either of these methods is much shorter than the earlier solutions; which of the two is the better in actual practice cannot be answered offhand. It may be remarked that the former uses almost entirely trigonometrical logarithms, which the latter does not, and this alone is a considerable advantage.

2.

All the equations and formulæ wanted in practice are collected later on. Here it is unnecessary to write down the six Lagrangian equations; as a type we confine our attention to that for the eccentricity

$$\frac{de}{dt} = \cos\phi\frac{d\phi}{dt} = \frac{a^2 n\cos\phi}{1+m}\Big[\sin v\mathrm{R} + (\cos v + \cos \mathrm{E})\mathrm{S}\Big]$$

We have to find that part of the values of R, S, and W which is independent of the time (or of the positions of the planets in their orbits).

As already stated, two integrations are required, one of which will be done mechanically. The above equation is therefore written

$$\left[\frac{d\phi}{dt}\right]_{00} = \frac{m_1 n}{1+m}\mathrm{M_E}\left[\cos\phi\sin\mathrm{E}\frac{a}{r}\mathrm{R}_0 - \left(\frac{3}{2}e - 2\cos\mathrm{E} + \frac{e}{2}\cos 2\mathrm{E}\right)\frac{a}{r}\mathrm{S}_0\right]$$

The mechanical integration need only be made on the quantities

$\frac{a}{r}\mathrm{R}_0$, $\frac{a}{r}\mathrm{S}_0$, and $\frac{a}{r}\mathrm{W}_0$, where

$$\frac{a}{r}\mathrm{R}_0 = \mathrm{A}_0^{(c)} + \frac{1}{2}\mathrm{A}_1^{(c)}\cos\mathrm{E} + \frac{1}{2}\mathrm{A}_1^{(s)}\sin\mathrm{E} + \frac{1}{2}\mathrm{A}_2^{(c)}\cos 2\mathrm{E} + \text{etc.}$$

with similar equations for $\frac{a}{r}\mathrm{S}_0$ and $\frac{a}{r}\mathrm{W}_0$.

Thus the final equations take the form

$$\left[\frac{d\phi}{dt}\right]_{00} = \frac{m_1 n}{1+m}\left[\frac{1}{2}\mathrm{A}_1^{(s)}\cos\phi - \frac{3}{2}e\mathrm{B}_0^{(c)} + \mathrm{B}_1^{(c)} - \frac{e}{4}\mathrm{B}_2^{(c)}\right]$$

The addendum to Hill's first paper may be referred to for further matter concerning these equations.

We have

$$\frac{a}{r}\mathrm{R}_0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^2}{r}\;\frac{xx_1 + yy_1 - r^2}{\Delta^3}(1 - e_1\cos\mathrm{E}_1)d\mathrm{E}_1$$

with corresponding expressions of the same form for $\frac{a}{r}\mathrm{S}_0$ and $\frac{a}{r}\mathrm{W}_0$.

Putting

$$\frac{a}{a_1} = \alpha \qquad\qquad r_0 = \alpha(1 - e\cos \mathrm{E}) = \frac{r}{a_1}$$

$$\begin{aligned} k\cos(v+\mathrm{K}) &= \mathrm{A}_c & k_1\cos\phi_1\sin(v+\mathrm{K}_1) &= \mathrm{A}_s \\ -k\sin(v+\mathrm{K}) &= \mathrm{B}_c & k_1\cos\phi_1\cos(v+\mathrm{K}_1) &= \mathrm{B}_s \\ \sin\Pi_1 &= \mathrm{C}_c & \cos\phi_1\cos\Pi_1 &= \mathrm{C}_s \end{aligned}$$

$$\begin{aligned} \mathrm{A}_0 &= 1 + r_0^2 + 2e_1 r_0 \mathrm{A}_c \qquad \text{Note, Hill's } \mathrm{A} = a_1^2\mathrm{A}_0, \text{ etc.} \\ \mathrm{B}_0\cos\epsilon &= r_0\mathrm{A}_c + e_1 \\ \mathrm{B}_0\sin\epsilon &= r_0\mathrm{A}_s \\ \mathrm{C}_0 &= e_1^2 \end{aligned}$$

$$\Delta_0^2 = \mathrm{A}_0 - 2\mathrm{B}_0\cos(\mathrm{E}_1 - \epsilon) + \mathrm{C}_0\cos^2\mathrm{E}_1$$

$$\text{Hill's } \Delta = a_1\Delta_0$$

then

$$\frac{a}{r}\mathrm{R}_0 = \frac{1}{2\pi}\int_0^{2\pi} \alpha^2 \frac{\mathrm{A}_c(\cos\mathrm{E}_1 - e_1) + \mathrm{A}_s\sin\mathrm{E}_1 - r_0}{\Delta_0^3}(1 - e_1\cos\mathrm{E}_1)d\mathrm{E}_1$$

$$\frac{a}{r}\mathrm{S}_0 = \frac{1}{2\pi}\int_0^{2\pi} \alpha^2 \frac{\mathrm{B}_c(\cos\mathrm{E}_1 - e_1) + \mathrm{B}_s\sin\mathrm{E}_1}{\Delta_0^3}(1 - e_1\cos\mathrm{E}_1)d\mathrm{E}_1$$

$$\frac{a}{r}\mathrm{W}_0 = \frac{1}{2\pi}\int_0^{2\pi} \alpha\sin\mathrm{J}\, r_0 \frac{\mathrm{C}_c(\cos\mathrm{E}_1 - e_1) + \mathrm{C}_s\sin\mathrm{E}_1}{\Delta_0^3}(1 - e_1\cos\mathrm{E}_1)d\mathrm{E}_1$$

If we remove the constant factors α^2, α^2, and $\alpha\sin\mathrm{J}$ respectively, the numerators of the fractions on the right may be written as

$$f - ge_1 + [g(1+e_1^2) - fe_1]\cos\mathrm{E}_1 + h\sin\mathrm{E}_1 - he_1\sin\mathrm{E}_1\cos\mathrm{E}_1 - ge_1\cos^2\mathrm{E}_1,$$

where f, g, and h have the respective values

	$\frac{1}{\alpha^2}\frac{a}{r_0}\mathrm{R}_0$	$\frac{1}{\alpha^2}\frac{a}{r_0}\mathrm{S}_0$	$\frac{1}{\alpha\sin\mathrm{J}}\frac{a}{r_0}\mathrm{W}_0 \times \frac{1}{r_0}$
f	$-r_0$	0	0
g	A_c	B_c	C_c
h	A_s	B_s	C_s

3.

Altering the variable from E_1 to T changes the equations for $\frac{1}{a^2}\frac{a}{r_0}R_0$, etc., to the form

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\Gamma_1+\Gamma_2\sin^2 T+\Gamma_3\cos^2 T}{\Delta_0^3}dT \quad . \quad . \quad (a)$$

wherein

$$\Delta_0^2 \text{ is now } G_1-G_2\sin^2 T+G_3\cos^2 T$$

but we may now omit the subscript $_0$ as unnecessary.

The elliptic integrals, using the limits o to $\frac{\pi}{2}$, are

$$\int_0^{\frac{\pi}{2}}\frac{dT}{\Delta^3}, \qquad \int_0^{\frac{\pi}{2}}\frac{\sin^2 T dT}{\Delta^3} \quad \text{and} \quad \int_0^{\frac{\pi}{2}}\frac{\cos^2 T dT}{\Delta^3}$$

The G's are the roots of the equation

$$x^3-(A-C)x^2+(B^2-AC)x+CB^2\sin^2\epsilon=0$$

or

$$x^3-k_1x^2+k_2x-k_3=0$$

Putting

$$e_1=G_1-\frac{k_1}{3}, \qquad e_2=G_2-\frac{k_1}{3}, \qquad e_3=-G_3-\frac{k_1}{3}$$

transforms

$$G_1-G_2\sin^2 T+G_3\cos^2 T \quad \text{into} \quad e_1-e_2\sin^2 T-e_3\cos^2 T$$

Introducing the Weierstrassian functions by means of

$$\sin^2 T=\frac{e_1-e_3}{e_2-e_3}\cdot\frac{s-e_2}{s-e_1} \qquad \left(\text{when } T=\begin{cases}0, & s=e_2\\ \frac{\pi}{2} & s=e_3\end{cases}\right.$$

gives

$$\cos^2 T=\frac{e_2-e_1}{e_2-e_3}\cdot\frac{s-e_3}{s-e_1}$$

and

$$\frac{dT}{ds}=\frac{1}{2}\sqrt{\frac{(e_1-e_3)(e_2-e_1)}{(s-e_2)(s-e_3)}}\cdot\frac{1}{s-e_1}$$

$$\Delta^2=(e_1-e_2)(e_1-e_3)/(s-e_1)$$

$$\frac{dT}{\Delta^3}=\frac{1}{2}\frac{s-e_1}{(e_1-e_2)(e_1-e_3)}\cdot\frac{ds}{\sqrt{(s-e_1)(s-e_2)(s-e_3)}}$$

Putting

$$s = \wp u \quad \text{and therefore} \quad ds = \wp' u du$$

and noting that

$$\wp' u = 2\sqrt{(\wp u - e_1)(\wp u - e_2)(\wp u - e_3)}$$

we have

$$\frac{dT}{\Delta^3} = \frac{\wp u - e_1}{(e_1 - e_2)(e_1 - e_3)} du$$

Having $\wp u = e_3$ when $u = \omega'$ and to e_2 when $u = \omega''$ or $\omega + \omega'$ and writing $\sqrt{G} = (e_1 - e_2)(e_1 - e_3)(e_2 - e_3)$ and noting that

$$- \wp u du = d \frac{\mathfrak{S}_1}{\mathfrak{S}} u$$

or

$$\int \wp u du = - \frac{\mathfrak{S}_1}{\mathfrak{S}} u$$

so that

$$\int_{\omega''}^{\omega'} \wp u du = \eta$$

we obtain

$$\int_0^{\frac{\pi}{2}} \frac{dT}{\Delta^3} = \frac{e_2 - e_3}{\sqrt{G}} (e_1 \omega + \eta)$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 T dT}{\Delta^3} = \frac{e_1 - e_3}{\sqrt{G}} (e_2 \omega + \eta) \quad . \quad . \quad . \quad (b)$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 T dT}{\Delta^3} = \frac{e_2 - e_1}{\sqrt{G}} (e_3 \omega + \eta)$$

The proofs of the above formulæ will be found in Padé's or Schwarz's paper already referred to. The italic G is not to be confounded with the three G roots.

The last set of formulæ correspond to Dr. Hill's $\frac{M}{m^3}$, $\frac{M}{m^3} \cos^2 \kappa$ and $\frac{M}{m^3} \sin^2 \kappa$ in the *Amer. Journ. of Math.*, t. xxiii. pp. 321–2.

Should the cubic equation be solved, the numerical value of these formulæ may be found as follows:—

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{dT}{\Delta^3} = \sqrt{\frac{e_2 - e_3}{G}}\ \frac{1}{4\sqrt{q}}\ \frac{1 + 3^2q^2 + 5^2q^6 + 7^2q^{12} + \text{etc.}}{(1 + q^2 + q^6 + q^{12} + \text{etc.})^3}$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2 T dT}{\Delta^3} = \sqrt{\frac{e_1 - e_3}{G}}\ 8q\ \frac{1 + 2^2q^3 + 3^2q^8 + 4^2q^{15} + \text{etc.}}{(1 + 2q + 2q^4 + 2q^9 + \text{etc.})^3}$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos^2 T dT}{\Delta^3} = \sqrt{\frac{e_1 - e_2}{G}}\ 8q\ \frac{1 - 2^2q^3 + 3^2q^8 - 4^2q^{15} + \text{etc.}}{(1 - 2q + 2q^4 - 2q^9 + \text{etc.})^3}$$

where q is to be derived as follows:—Let

$$\frac{e_1 - e_2}{e_1 - e_3} = \cos^2\theta \text{ and } \tfrac{1}{2}l = \left[\frac{\sin\frac{\theta}{2}}{1 + \sqrt{\cos\theta}}\right]^2$$

then

$$q = \tfrac{1}{2}l + 2(\tfrac{1}{2}l)^5 + 15(\tfrac{1}{2}l)^9 + 150(\tfrac{1}{2}l)^{13} + \text{etc.}$$

or in practice

$$\log q = \log \tfrac{1}{2}l + [9{\cdot}9388]\,(\tfrac{1}{2}l)^4$$

(see *M. N.*, R.A.S., vol. lxii. p. 494).

All these formulæ are extraordinarily convergent. They can also be derived from the formulæ given on p. 333 of Hill's paper (*A.J.*, No. 511) in virtue of the relations

$$k = 4\sqrt{q}\left[\frac{1 + q^2 + q^6 + q^{12} + \text{ etc.}}{1 + 2q + 2q^4 + 2q^9 + \text{ etc.}}\right]^2 \qquad k_1 = \left[\frac{1 - 2q + 2q^4 - 2q^9 + \text{ etc.}}{1 + 2q + 2q^4 + 2q^9 + \text{ etc.}}\right]^2$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{dT}{\sqrt{1 - k^2\sin^2 T}} = (1 + 2q + 2q^4 + 2q^9 + \text{ etc.})^2$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2 T}\,dT = \left[\frac{1 + 3^2q^2 + 5^2q^6 + 7^2q^{12} + \text{ etc.}}{1 - 3q^2 + 5q^6 - 7q^{12} + \text{ etc.}}\right] \times \sqrt{k_1}$$

the last formula being one of some half dozen representations of Legendre's E by means of Jacobi's q function.

4.

Introducing the equations (b) into the equation (a) and putting

$$\mathrm{M} = (e_2 - e_3)\Gamma_1 + (e_1 - e_3)\Gamma_2 + (e_2 - e_1)\Gamma_3$$
$$\mathrm{N} = e_1(e_2 - e_3)\Gamma_1 + e_2(e_1 - e_3)\Gamma_2 + e_3(e_2 - e_1)\Gamma_3$$

enables us to write

$$\frac{1}{a^2}\frac{a}{r}\mathrm{R}_0 = \frac{2}{\pi}\frac{1}{\sqrt{G}}\Big[\mathrm{M}\eta + \mathrm{N}\omega\Big] \quad . \quad . \quad . \quad (c)$$

with similar simple expressions for the two other components. If M and N are written in determinant form they are

$$\mathrm{M} = \begin{vmatrix} \Gamma_1 & -\Gamma_2 & -\Gamma_3 \\ e_1 & e_2 & e_3 \\ 1 & 1 & 1 \end{vmatrix}. \qquad \mathrm{N} = \begin{vmatrix} e_1\Gamma_1 & -e_2\Gamma_2 & -e_3\Gamma_3 \\ e_1 & e_2 & e_3 \\ 1 & 1 & 1 \end{vmatrix}$$

5.

We now proceed to the elimination of the roots of the cubic (e_1, e_2 and e_3) by means of the invariants g_2 and g_3. The following formulæ are collected here for easy reference:—

$$\mathrm{G}_1 + \mathrm{G}_2 - \mathrm{G}_3 = k_1. \quad \mathrm{G}_1\mathrm{G}_2 - \mathrm{G}_1\mathrm{G}_3 - \mathrm{G}_2\mathrm{G}_3 = k_2. \quad -\mathrm{G}_1\mathrm{G}_2\mathrm{G}_3 = k_3$$
$$\tfrac{3}{4}g_2 = \lambda = k_1^2 - 3k_2 = -3(e_1e_2 + e_1e_3 + e_2e_3)$$
$$\tfrac{1}{4}g_3 = \tfrac{1}{27}(2k_1^3 - 9k_1k_2 + 27k_3) = e_1e_2e_3$$
$$16G = g_2^3 - 27g_3^2 = 16(e_1 - e_2)(e_1 - e_3)(e_2 - e_3)$$

or for the last

$$\sqrt{G} = \tfrac{1}{4}g_3 \times \begin{vmatrix} 1, & 1, & 1 \\ \dfrac{1}{e_1} & \dfrac{1}{e_2} & \dfrac{1}{e_3} \\ e_1 & e_2 & e_3 \end{vmatrix} = \tfrac{1}{4}g_3 \times \begin{vmatrix} 1, & 1, & 1 \\ \dfrac{1}{e_1} & \dfrac{1}{e_2} & \dfrac{1}{e_3} \\ \dfrac{e_2e_3}{e_1} & \dfrac{e_1e_3}{e_2} & \dfrac{e_1e_2}{e_3} \end{vmatrix}$$

We have now four determinants; let us multiply the first by the third, and then the second by the fourth. The first result is written at length,

$$\mathrm{M}\sqrt{G} = \begin{vmatrix} \Gamma_1 - \Gamma_2 - \Gamma_3, & e_2e_3\Gamma_1 - e_1e_3\Gamma_2 - e_1e_2\Gamma_3, & e_1\Gamma_1 - e_2\Gamma_2 - e_3\Gamma_3 \\ 0 & 3e_1e_2e_3 & -2(e_1e_2 + e_1e_3 + e_2e_3) \\ 3 & e_1e_2 + e_1e_3 + e_2e_3 & 0 \end{vmatrix}$$

To abbreviate put

$$\begin{aligned} \Gamma_1 - \quad \Gamma_2 - \quad \Gamma_3 &= \psi \\ e_2e_3\Gamma_1 - e_1e_3\Gamma_2 - e_1e_2\Gamma_3 &= \chi \quad . \quad . \quad . \quad (d) \\ e_1\Gamma_1 - \quad e_2\Gamma_2 - \quad e_3\Gamma_3 &= \phi \end{aligned}$$

and replace therein the constituents in e_1, e_2, and e_3 by their appropriate equivalents in g_2 and g_3; then expanding we obtain finally

$$\mathrm{M}\sqrt{G} = \psi\tfrac{1}{8}g_2^2 + \tfrac{3}{2}g_2\chi - \tfrac{9}{4}g_3\phi$$

A similar process gives

$$\mathrm{N}\sqrt{G} = -\psi\tfrac{3}{16}g_2g_3 - \tfrac{9}{4}g_3\chi + \tfrac{1}{8}g_2^2\phi$$

so that (*c*) becomes

$$\frac{1}{a^2}\frac{a}{r}R_0=\frac{2}{\pi}\frac{1}{\sqrt{G}}\left[\tfrac{1}{2}(g_2\eta-\tfrac{3}{2}g_3\omega)(\tfrac{1}{4}g_2\psi+3\chi)+\tfrac{1}{4}(\tfrac{1}{2}g_2^2\omega-9g_3\eta)\phi\right]$$

with similar equations for S_0 and W_0.

6.

Bruns first pointed out that the periods of the elliptic functions could be found without a knowledge of the three roots (H. Bruns, *Perioden der Ellip. Integrale,* Dorpat, 1875).

This remarkable investigation does not deserve the neglect it has received at the hands of English mathematicians. Bruns does not adapt his formulæ for numerical use, and I have found it better to adopt another absolute invariant. If we write

$$\cos\iota=3\sqrt{3}\frac{g_3}{g_2^{\frac{3}{2}}}\qquad(0°<\iota<180°)$$

we have

$$\frac{2}{\pi}\frac{1}{\sqrt{G}}\frac{1}{2}\left(g_2\eta-\frac{3}{2}g_3\omega\right)=\frac{5}{8}\frac{1}{\lambda^{\frac{1}{4}}}\frac{1}{\cos^2\frac{\iota}{2}}F\left(\frac{1}{6},\frac{5}{6},2.\sin^2\frac{\iota}{2}\right)$$

$$\frac{2}{\pi}\frac{1}{\sqrt{G}}\frac{1}{4}\left(\frac{1}{2}g_2^2\omega-9g_3\eta\right)=\frac{7}{8}\frac{1}{\lambda^{\frac{1}{4}}}\frac{\lambda^{\frac{1}{2}}}{\cos^2\frac{\iota}{2}}F\left(-\frac{1}{6},\frac{7}{6},2.\sin^2\frac{\iota}{2}\right)$$

Hence

$$\frac{1}{a^2}\frac{a}{r}R_0=\frac{1}{8}\frac{1}{\cos^2\frac{\iota}{2}}\frac{1}{\lambda^{\frac{1}{4}}}\left[\left(\frac{1}{4}g_2\psi+3\chi\right)5F\left(\frac{1}{6}\right)+\lambda^{\frac{1}{2}}\phi 7F\left(-\frac{1}{6}\right)\right]$$

with similar equations for S_0 and W_0. The abbreviations $F\left(\frac{1}{6}\right)$ and $F\left(-\frac{1}{6}\right)$ require no explanation.

7.

It remains to find the values of ψ, χ, and ϕ. Replacing the *e*'s by the G's in (*d*) we have

$$\psi=\Gamma_1-\Gamma_2-\Gamma_3.$$

$$\phi=\left(G_1-\frac{k_1}{3}\right)\Gamma_1-\left(G_2-\frac{k_1}{3}\right)\Gamma_2+\left(G_3+\frac{k_1}{3}\right)\Gamma_3=G_1\Gamma_1-G_2\Gamma_2+G_3\Gamma_3-\frac{k_1}{3}\psi$$

$$\chi=-G_2G_3\Gamma_1+G_1G_3\Gamma_2-G_1G_2\Gamma_3+\frac{k_1}{3}\phi-\left(\frac{k_1}{3}\right)^2\psi$$

Hill and Arndt give the formulæ for these quantities in G which are as follows:—

	$\frac{1}{a^2}\frac{a}{r}R_0$	$\frac{1}{a^2}\frac{a}{r}S_0$	$\frac{1}{a \sin J}\frac{a}{r}W_0$
$\Gamma_1 - \Gamma_2 - \Gamma_3 = \psi$	$-r_0$	0	0
$G_1\Gamma_1 - G_2\Gamma_2 + G_3\Gamma_3 = \phi + \frac{k_1}{3}\psi$	$-r_0(A_0 - e_1 B_0 \cos \epsilon) + A_c D + A_s B_0 \sin \epsilon$	$0 + B_c D + B_s B_0 \sin \epsilon$	$0 + C_c D + C_s B_0 \sin \epsilon$
$-G_2G_3\Gamma_1 + G_1G_3\Gamma_2 - G_1G_2\Gamma_3 = \chi - \frac{k_1}{3}\phi + \frac{k_1^2}{9}\psi$	$-e_1 r_0 B_0 \sin \epsilon (A_s A_c - A_c A_s) = 0$	$-e_1 r_0 B_0 \sin \epsilon (B_s A_c - B_c A_s) = -e_1 r_0 B_0 \sin \epsilon \cos \phi_1 \cos J$	$-e_1 r_0 B_0 \sin \epsilon (C_s A_c - C_c A_s) = -e_1 r_0 B_0 \sin \epsilon \cos \phi_1 \cos(\nu + \Pi)$

[where $D = (1 + e_1^2)B_0 \cos \epsilon - e_1(A_0 + C_0)$].

These formulæ simplify in use. Working formulæ and two numerical applications follow.

Working Formulæ.

9.

Let

a = semi-axis major corrected for the constant part of its perturbations
n = mean motion
$e = \sin\phi$ = eccentricity
π = long. of perihelion from equinox
i = inclination of orbit to ecliptic
Ω = long. of ascending node
χ = long. of perihelion from a fixed point in orbit
$\omega = \pi - \Omega$
r = radius-vector
E = excentric anomaly
ν = true anomaly
m = mass (Sun's mass unity)

of the disturbed body; the addition of a suffix ($_1$) indicates similar quantities appertaining to the disturbing body.

10.

$$\begin{aligned}
-\sin\tfrac{1}{2}J\sin\tfrac{1}{2}(\Phi+\Psi) &= \sin\tfrac{1}{2}(\Omega_1-\Omega)\sin\tfrac{1}{2}(i_1+i)\\
-\sin\tfrac{1}{2}J\cos\tfrac{1}{2}(\Phi+\Psi) &= \cos\tfrac{1}{2}(\Omega_1-\Omega)\sin\tfrac{1}{2}(i_1-i)\\
\cos\tfrac{1}{2}J\sin\tfrac{1}{2}(\Phi-\Psi) &= \sin\tfrac{1}{2}(\Omega_1-\Omega)\cos\tfrac{1}{2}(i_1+i)\\
\cos\tfrac{1}{2}J\cos\tfrac{1}{2}(\Phi-\Psi) &= \cos\tfrac{1}{2}(\Omega_1-\Omega)\cos\tfrac{1}{2}(i_1-i)
\end{aligned}$$

check formulæ

$$\begin{aligned}
\cos p\sin q &= \sin i_1\cos(\Omega_1-\Omega)\\
\cos p\cos q &= \cos i_1\\
\cos p\sin r &= -\cos i_1\sin(\Omega_1-\Omega)\\
\cos p\cos r &= \cos(\Omega_1-\Omega)\\
\sin p &= -\sin i_1\sin(\Omega_1-\Omega)
\end{aligned}$$

then

$$\begin{aligned}
\sin J\sin\Phi &= \sin p\\
\sin J\cos\Phi &= \cos p\sin(i-q)\\
\sin J\sin(\Psi-r) &= \sin p\cos(i-q)\\
\sin J\cos(\Psi-r) &= \sin(i-q)\\
\cos J &= \cos p\cos(i-q)
\end{aligned}$$

$$\Pi = \omega - \Phi. \qquad \Pi_1 = \omega_1 - \Psi$$

$$k\cos(K-\Pi) = \cos\Pi_1. \qquad k_1\cos(K_1-\Pi) = \cos\Pi_1\cos J$$
$$k\sin(K-\Pi) = -\sin\Pi_1\cos J. \qquad k_1\sin(K_1-\Pi) = -\sin\Pi_1$$
$$\text{check: } k^2+k_1^2 = 1+\cos^2 J$$

When the Earth is the disturbing planet, these equations simplify to

$$\begin{aligned}
J &= i\\
\Phi &= 0\\
\Pi &= \pi-\Omega\\
\Pi_1 &= \pi_1-\Omega
\end{aligned}$$

and when the Earth is the disturbed planet, to

$$J = i_1$$
$$\Psi = 180°$$
$$\Pi = 180° + \pi - \Omega_1$$
$$\Pi_1 = 180° + \pi_1 - \Omega_1$$

II.

The various functions of E which follow should now be computed for $4j$ equal parts of the circumference. The larger the number of parts is, the more exact will be the results of the mechanical integration, but j should not be taken larger than necessary. A measure of the accuracy attained is given by adding the odd values and the even values of each of the quantities: if these differ, more values of E are required. Reference to Hill's original example or to those of later computers will assist the judgment.

Take $\alpha = a/a_1$, then omitting the suffix $_0$ used in the demonstration, when no confusion will be created thereby, we have

$$r_0 \cos v = \alpha(\cos E - e). \qquad r_0 \sin v = \alpha \cos \phi \sin E$$
$$r_0 = \alpha(1 - e \cos E). \text{ (check)}$$
$$k \cos (v + K) = A_c \qquad k_1 \cos \phi_1 \sin (v + K_1) = A_s$$
$$-k \sin (v + K) = B_c \qquad k_1 \cos \phi_1 \cos (v + K_1) = B_s$$
$$A = 1 + r_0^2 + 2e_1 r_0 A_c.$$
$$B \cos \epsilon = r_0 A_c + e_1. \qquad B \sin \epsilon = r_0 A_s$$
$$k_1 = A - e_1^2. \qquad k_2 = B_2 - Ae_1^2. \qquad -k_3 = e_1^2 B^2 \sin^2 \epsilon$$
$$\tfrac{3}{4} g_2 = \lambda = k_1^2 - 3k_2 \qquad g_3 = \tfrac{4}{27}(2k_1^3 - 9k_1 k_2 + 27k_3)$$
$$\cos \iota = \frac{\sqrt{27}\, g_3}{g_2^{\frac{3}{2}}}. \qquad (0° < \iota < 180°)$$
$$D = (1 + e_1^2) B \cos \epsilon - e_1(A + e_1^2)$$
$$\phi_R = -r_0\left(A - e_1 B \cos \epsilon - \frac{k_1}{3}\right) + A_c D + A_s B \sin \epsilon$$
$$\phi_S = \qquad + B_c D + B_s B \sin \epsilon$$
$$\phi_W = \qquad + \sin \Pi_1 D + \cos \phi_1 \cos \Pi_1 B \sin \epsilon$$
$$\chi_R - \frac{k_1}{3}\phi_R - \frac{k_1^2}{9} r_0 = 0$$
$$\chi_S - \frac{k_1}{3}\phi_S \qquad = -e_1 r_0 B \sin \epsilon \cos \phi_1 \cos J$$
$$\chi_W - \frac{k_1}{3}\phi_W \qquad = -e_1 r_0 B \sin \epsilon \cos \phi_1 \cos (v + \Pi)$$

With the aid of the tables or formulæ compute

$$\frac{5}{8}\frac{1}{\cos^2\frac{\iota}{2}}\,F\left(1/6\,,\,5/6\,,\,2\,,\,\sin^2\frac{\iota}{2}\right)=F_A$$

and

$$\frac{7}{8}\frac{1}{\cos^2\frac{\iota}{2}}F\left(-1/6\,,\,7/6\,,\,2\,,\,\sin^2\frac{\iota}{2}\right)=F_B$$

Then

$$\frac{1}{a_2}\frac{a}{r}R_0=\left(3\chi-\frac{1}{4}g_2r_0\right)\frac{F_B}{\lambda^{\frac{7}{4}}}+\frac{F_A}{\lambda^{\frac{5}{4}}}$$

$$\frac{1}{a_2}\frac{a}{r}S_0=(\quad 3\chi\quad)\text{ ,, ,, ,,}$$

$$\frac{1}{a}\frac{1}{\sin J}\frac{a}{r}W_0=[(\quad 3\chi\quad)\text{ ,, ,, ,,}]r_0$$

12.

With the $4j$ values of these three functions, which may be designated

$$\begin{array}{ll} R^{(0)},\ R^{(1)},\ R^{(2)},\ \ldots\ldots & R^{(4j-1)} \\ S^{(0)},\ S^{(1)},\ \ldots\ldots\ldots & S^{(4j-1)} \\ W^{(0)},\ W^{(1)},\ \ldots\ldots\ldots & W^{(4j-1)} \end{array}$$

compute

$$A_0^{(c)}=\frac{1}{4j}[R^{(0)}+R^{(1)}+R^{(2)}\ \ldots\ldots\ R^{(4j-1)}]$$

$$\frac{1}{2}A_1^{(c)}=\frac{1}{4j}\left[R^{(0)}+R^{(1)}\cos\frac{1}{j}\frac{\pi}{2}+R^{(2)}\cos\frac{2}{j}\frac{\pi}{2}\ \ldots\ldots\ +R^{(4j-1)}\cos\frac{4j-1}{j}\frac{\pi}{2}\right]$$

$$\frac{1}{2}A^{(s)}=\frac{1}{4j}\left[\qquad R^{(1)}\sin\frac{1}{j}\frac{\pi}{2}+R^{(2)}\sin\frac{2}{j}\frac{\pi}{2}\ \ldots\ldots\ +R^{(4j-1)}\sin\frac{4j-1}{j}\frac{\pi}{2}\right]$$

$$\frac{1}{2}A_2^{(c)}=\frac{1}{4j}\left[R^{(0)}+R^{(1)}\cos\frac{2}{j}\frac{\pi}{2}+R^{(2)}\cos\frac{4}{j}\frac{\pi}{2}\ \ldots\ldots\ +R^{(4j-1)}\cos\frac{2(4j-1)}{j}\frac{\pi}{2}\right]$$

$$\frac{1}{2}A_2^{(s)}=\frac{1}{4j}\left[\qquad R^{(1)}\sin\frac{2}{j}\frac{\pi}{2}+R^{(2)}\sin\frac{4}{j}\frac{\pi}{2}\ \ldots\ldots\ +R^{(4j-1)}\sin\frac{2(4j-1)}{j}\frac{\pi}{2}\right]$$

with similar equations for $B_0^{(c)}$ $B_1^{(c)}$ .... and $C_0^{(c)}$ $C_1^{(c)}$ and $C_1^{(s)}$.
(Note :— $A_2^{(s)}$ is not required.)

For any special case of j these equations become greatly simplified: thus for $j=2$ we have

$$A_0^{(c)}=\tfrac{1}{8}[R^{(0)}+R^{(1)}+R^{(2)}\ \ldots\ldots\ +R^{(7)}]$$

$$\tfrac{1}{2}A_1^{(c)}=\tfrac{1}{8}[R^{(0)}-R^{(4)}+(R^{(1)}-R^{(3)}-R^{(5)}+R^{(7)})\cos 45^\circ]$$

$$\tfrac{1}{2}A_1^{(s)}=\tfrac{1}{8}[R^{(2)}-R^{(6)}+(R^{(1)}+R^{(3)}-R^{(5)}-R^{(7)})\sin 45^\circ]$$

$$\tfrac{1}{2}A_2^{(c)}=\tfrac{1}{8}[R^{(0)}-R^{(2)}+R^{(4)}-R^{(6)}]$$

$$\tfrac{1}{2}A_2^{(s)}=\tfrac{1}{8}[R^{(1)}-R^{(3)}+R^{(4)}-R^{(6)}]$$

and so on. For other values of j the equations are easily formed, or may be found in Hansen's *Auseinandersetzung* or elsewhere.

The equation

$$\sin\phi \tfrac{1}{2} A_1^{(s)} + \cos\phi B_0^{(c)} = 0$$

affords a useful check.

13.

The final equations are

$$e\left[\frac{d\phi}{dt}\right]_{00} = \frac{m_1 n}{1+m}\, a^2 \left[-\left(1+\frac{e^2}{2}\right)B_0^{(c)} - 2e\frac{1}{2}B_1^{(c)} + \frac{e^2}{2}\frac{1}{2}B_2^{(c)}\right]$$

$$e\left[\frac{d\chi}{dt}\right]_{00} = \frac{m_1 n}{1+m}\, a_2 \left[eA_0^{(c)}\cos\phi - \frac{1}{2}A_1^{(c)}\cos\phi + (2-e^2)\frac{1}{2}B_1^{(s)} - \frac{e}{2}\frac{1}{2}B_2^{(s)}\right]$$

$$\left[\frac{di}{dt}\right]_{00} = \frac{m_1 n}{1+m}\, a \sin J \cos\omega \left[\left(\frac{1}{2}C_1^{(c)} - eC_0^{(c)}\right)\sec\phi - \frac{1}{2}C_1^{(s)}\tan\omega\right]$$

$$\sin i\left[\frac{d\Omega}{dt}\right]_{00} = \frac{m_1 n}{1+m}\, a \sin J \cos\omega \left[\left(\frac{1}{2}C_1^{(c)} - eC_0^{(c)}\right)\sec\phi\tan\omega + \frac{1}{2}C_1^{(s)}\right]$$

$$\left[-2\frac{r}{a}R_0\right]_{00} = a^2\left[-(2+e^2)A_0^{(c)} + 4e\frac{1}{2}A_1^{(c)} - e^2\frac{1}{2}A_2^{(c)}\right]$$

Calculation of $F\left(\frac{1}{6}, \frac{5}{6}, 2, \sin^2\frac{\iota}{2}\right)$ and $F\left(-\frac{1}{6}, \frac{7}{6}, 2, \sin^2\frac{\iota}{2}\right)$

14.

When ι is less than 90° the calculation of these series is not long, but if an accuracy of two units in the 7th place of the decimals of the logarithms is sufficient, the use of series can be avoided. This accuracy is indeed greater than either our knowledge of the planetary masses or the precision of observations requires. Besides these, the change in the secular perturbations due to higher powers of the masses than the first is of a more important character.

The approximate formulæ are

$$\log F\left(\frac{1}{6}, \frac{5}{6}, 2.\ \sin^2\frac{\iota}{2}\right) = \log\left\{1 + \frac{\frac{5}{72}\sin^2\frac{\iota}{2}}{\left(\sqrt{1-\frac{2}{3}\sin^2\frac{\iota}{2}}\right)^{1+\frac{5}{72}}}\right\} + [6{\cdot}0049]\left(\frac{\iota^\circ}{100}\right)^{\frac{100}{12}}$$

$$\log F\left(-\frac{1}{6}, \frac{7}{6}, 2.\ \sin^2\frac{\iota}{2}\right) = \log\left\{1 - \frac{\frac{7}{72}\sin^2\frac{\iota}{2}}{\left(\sqrt{1-\frac{2}{3}\sin^2\frac{\iota}{2}}\right)^{1-\frac{7}{72}}}\right\} - [6{\cdot}1011]\left(\frac{\iota^\circ}{100}\right)^{\frac{100}{12}}$$

The two series are inert :—

ι	$F\left(\frac{1}{6}, \frac{5}{6}, 2, \sin^2\frac{\iota}{2}\right)$	$F\left(-\frac{1}{6}, \frac{7}{6}, 2, \sin^2\frac{\iota}{2}\right)$
0°	1·0000000	1·0000000
90°	0·9415120	1·0432298
180°	$0·8185111 = \frac{2}{\pi}\frac{9}{7}$	$1·1459156 = \frac{2}{\pi}\frac{9}{5}$

If ι exceeds 90° the best procedure seems to be as follows :—

$$\text{Take} \qquad \iota_1 = 180° - \iota,$$

then proceeding at once to the actual quantities wanted and writing

$$H = \frac{2}{\pi}\frac{108}{\sin^2\frac{\iota_1}{2}}\frac{1}{\left[7F\left(-\frac{1}{6}, \frac{7}{6}, 2, \sin^2\frac{\iota_1}{2}\right) + 5\cos\iota_1 F\left(\frac{1}{6}, \frac{5}{6}, 2, \sin^2\frac{\iota_1}{2}\right)\right]}$$

we have

$$\frac{5}{\cos^2\frac{\iota}{2}} F\left(\frac{1}{6}, \frac{5}{6}, 2, \sin^2\frac{\iota}{2}\right) = H - \frac{1}{\pi}\log_e\frac{1}{q}\frac{5}{\cos^2\frac{\iota_1}{2}} F\left(\frac{1}{6}, \frac{5}{6}, 2, \sin^2\frac{\iota_1}{2}\right)$$

$$\frac{7}{\cos^2\frac{\iota}{2}} F\left(-\frac{1}{6}, \frac{7}{6}, 2, \sin^2\frac{\iota}{2}\right) = H\cos\iota_1 + \frac{1}{\pi}\log_e\frac{1}{q}\frac{7}{\cos^2\frac{\iota_1}{2}} F\left(-\frac{1}{6}, \frac{7}{6}, 2, \sin^2\frac{\iota_1}{2}\right)$$

To compute q put

$$\frac{1}{\sqrt{3}}\tan\frac{\iota_1}{3} = \cos 2\gamma$$

and $\qquad \sqrt{\tan\gamma} = \cos 2\lambda$

then $\qquad l = \tan^2\lambda$

and $\qquad q = \frac{1}{2}l + 2(\frac{1}{2}l)^5 + \text{etc.}$

In practice it may be assumed that $\log q = \log \frac{1}{2}l$. This assumption is in defect by the following quantities :—

ι_1	Correction to $\log \frac{1}{2}l$ in units of the 7th place of decimals,
0° – 40°	+0
40°	1
50°	2
60°	4
70°	9
80°	17
90°	+30

As $\log_e \frac{1}{q}$ is the natural log of $\frac{1}{q}$, we have in ordinary logarithms—

$$\log\left(\frac{1}{\pi}\log_e \frac{1}{q}\right) = \log \text{ of } \log \frac{1}{q} + 9{\cdot}8650658$$

15.

Numerical Examples.

In the *Astron. Nachr.*, No. 4068, Dr Arthur B. Turner has computed the secular perturbations of Mars by Jupiter using Arndt's formulæ.

The preceding formulæ are applied to this case for the case of $E = 0°$. We have $J = 1°26'$, $\Pi = 149°{\cdot}8$, $\Pi_1 = 188°{\cdot}4$, K and $K_1 = 321°{\cdot}4$, k and $k_1 = 1$, $\log r_0 = 9{\cdot}4241$, $\nu = 0°$, $\log A_c = 9{\cdot}893$, $\log B_c = 9{\cdot}795$, $\log A_s = 9{\cdot}794n$, $\log B_s = 9{\cdot}892$, $A_0 = 1{\cdot}09$, $\log B_0 \sin \epsilon = 9{\cdot}219$, $k_1 = 1{\cdot}088$, $k_2 = 0{\cdot}090$, $k_3 = -0{\cdot}000$, $g_2 = 1{\cdot}218$, $g_3 = 0{\cdot}251$, $\iota = 14°20'$, $F(\frac{1}{6}) = 0{\cdot}998$, $F(-\frac{1}{6}) = 1{\cdot}001$, $D = 0{\cdot}204$, and

	R_0	S_0	W_0
ϕ	0·072	0·002	0·134
χ	0·061	0·001	0·050

With these figures Turner's results are reproduced.

The next example is taken from Herr W. Dziewulski's paper on the secular perturbations of Eros by Mars. In the present position of the orbits for $E = 120°$, the modular angle $\theta = 58°{\cdot}9$. In the notation of the present paper we have $J = 11°{\cdot}5$, $\Pi = 6°{\cdot}7$, $\Pi_1 = 219°{\cdot}6$, $K = 147°{\cdot}6$, $K_1 = 146°{\cdot}5$, $\log k = 9{\cdot}996$, $\log k_1 = 9{\cdot}995$, $\log r_0 = 0{\cdot}0268$, $\nu = 130°{\cdot}6$, $\log A_c = 9{\cdot}152$, $\log B_c = 9{\cdot}992$, $\log A_s = 9{\cdot}990n$, $\log B_s = 9{\cdot}084$, $A_0 = 2{\cdot}16$, $\log B_0 \sin \epsilon = 0{\cdot}016n$, $k_1 = 2{\cdot}151$, $k_2 = 1{\cdot}120$, $k_3 = -0{\cdot}009$, $g_2 = 1{\cdot}690$, $g_3 = -0{\cdot}300$, $\iota = 135°{\cdot}1$, $\iota_1 = 44°{\cdot}9$, $\gamma = 40°{\cdot}6$, $\lambda = 11°{\cdot}1$, $\log \frac{1}{2}l = 8{\cdot}288$, $\log \frac{1}{\pi} \log_e \frac{1}{q} = 0{\cdot}099$, $F_1(\frac{1}{6}) = 0{\cdot}985$, $F_1(-\frac{1}{6}) = 1{\cdot}011$, $F(\frac{1}{6}) = 0{\cdot}881$, $F(-\frac{1}{6}) = 1{\cdot}102$, $D = 0{\cdot}044$, and

	R_0	S_0	W_0
ϕ	−0·490	−0·083	0·763
χ	0·195	0·041	0·472

leading to Dziewulski's results.

[For Tables, see paper by Mr Robbins.]

Johannesburg.

Tables for the Application of Mr Innes's Method.

By Frank Robbins.

As an appendix to the preceding paper on "The Computation of Secular Perturbations," the author asked me to compute for each degree of the quadrant the logarithmic values (base 10) of the two functions of *iota* required for the convenient application of his method.

In the hypergeometric series F $(\alpha\,\beta\,\gamma\,x)$ in the first case

$$\alpha \text{ has the value } -\frac{1}{6} \qquad \beta=\frac{7}{6} \qquad \gamma=2 \qquad x=\sin^2\frac{\iota}{2}$$

and in the second case

$$\alpha=\frac{1}{6} \qquad \beta=\frac{5}{6} \qquad \gamma=2 \qquad x=\sin^2\frac{\iota}{2}$$

For convenience of designation the tables are headed *Minus* F and *Plus* F according to the sign of α.

Vega's (1794) ten-figure logarithms, corrected by collation with the copy in use at H.M. *Nautical Almanac* Office, were used, and the natural values of the individual terms were taken out to ten places of decimals. These were obtained in duplicate for each end of the quadrant, and the whole were examined by differencing to the sixth order. Lastly, the seven-figure logarithms of the functions were taken from the eight-figure table of the Service Géographique de l'Armée (Paris, 1891), reference being made to Vega where the eighth figure was approximately five.

The log *Minus* F has been increased by 10 as customary, to avoid the inconvenience of printing negative characteristics.

The whole has been examined by Mr J. Abner Sprigge, of H.M. *Nautical Almanac* Office, so as to make it possible to use the tables with confidence in their accuracy to the seventh place.

(Iota).	Log plus F.	Δ_1	Δ_2	Log minus F.	Δ_1	Δ_2
1°	0·0000023			9·9999968		
		+ 69			− 97	
2	0092		+46	9871		−63
		115			160	
3	0207		45	9711		65
		160			225	
4	0367		47	9486		65
		207			290	
5	0574		46	9196		64
		253			354	
6	0827		46	8842		64
		299			418	
7	1126		45	8424		64
		344			482	
8	0·0001470		+46	9·9997942		−65
		+390			−547	